Characterizations of Strong Metric Regularity in Nonlinear Optimization

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Optimization is not a Singular Event

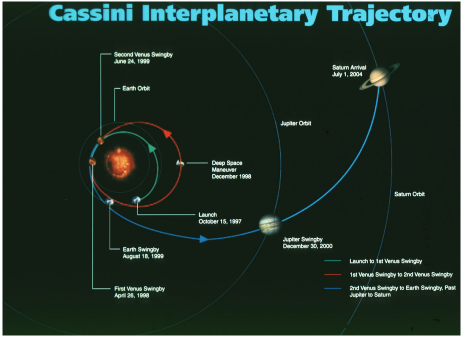


Figure: Interplanetary maneuver.
(Source: NASA)

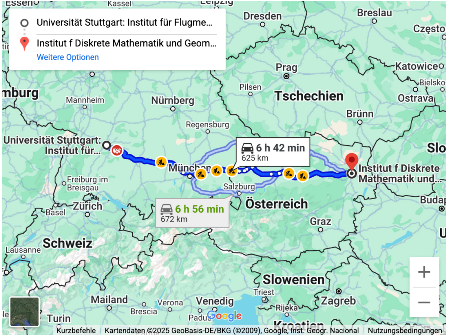
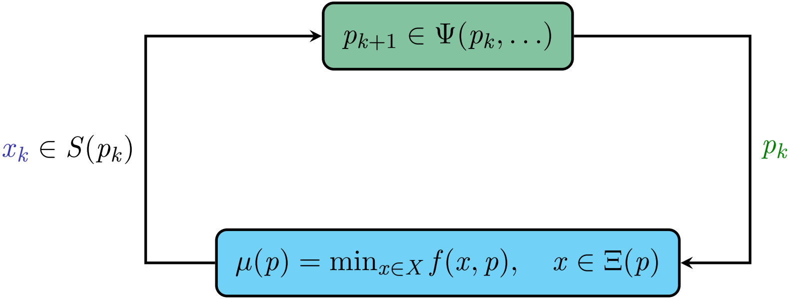


Figure: Planetary maneuver.
(Source: Google Maps)

Optimization in (Feedback) Loops

- dynamical system (MPC)
- central unit (distributed OP)
- hyper-parameter optimization
- any other upper-level OP



- optimal-control problem
- convex / unconstrained OP
- neural network training
- any other lower-level OP

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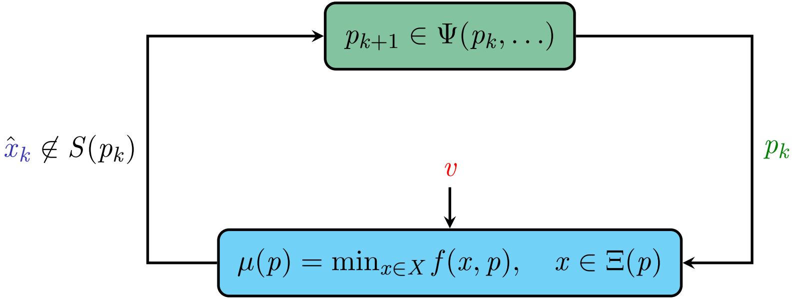

- optimal-control problem
- convex / unconstrained OP
- neural network training
- any other lower-level OP

Examples

- Model predictive control

$$
\begin{aligned}
u_k &\in \arg\min_u \mathrm{OCP}(u, x_k) \\
x_{k+1} &= f(x_k, u_k)
\end{aligned}
$$

- Hyper-parameter training

$$
\begin{aligned}
\theta_k &\in \arg\min_\theta \mathrm{LSQ}(\theta, p_k) \\
p_{k+1} &= p_k - \partial\,\mathrm{CVA}(p_k, \theta_k)
\end{aligned}
$$

- Augmented Lagrangian Method

$$
\begin{aligned}
x_k &\in \arg\min_x L_\varrho(x, y_k) \\
y_{k+1} &= y_k - \varrho f(x_k)
\end{aligned}
$$

Input-to-state Stability

Question:

- How does a disturbance affect the interconnected dynamics?

Definition

A dynamic system (e.g., $x_{k+1} = \phi(x_k, v)$) is *input-to-state stable* (ISS) iff

0-GAS: the equilibrium $\bar{x}$ is globally asymptotically stable for $v \equiv 0$;

AG: $\exists \gamma \in \mathcal{K},\ \forall x_0,\ \forall v \in \ell_\infty,\ \limsup_{N \to \infty} \|x_N\| \leq \gamma(\sup_{k \geq 0} \|v_k\|)$.

Interconnections of ISS Systems

- Let ψ_1, ψ_2 be ISS with gains $\gamma_1, \gamma_2 \in \mathcal{K}$

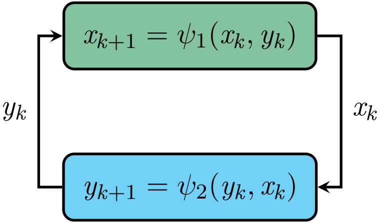

Figure: ISS if $\gamma_1 \circ \gamma_2 \prec \mathrm{id}$

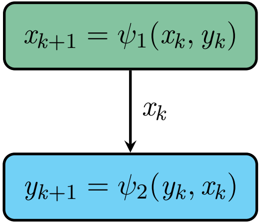

Figure: always ISS

ISS of Optimization Loops

If the optimization algorithm is (locally) ISS, we can prove...

- that an *MPC feedback asymptotically stabilizes* with a finite number of iterations [LMNK20]
- that a *gradient-based bilevel scheme converges* with inexact lower-level solutions and gradients [CK23], even without differentiability in the lower level [CK24a]

ISS of Optimization Algorithms

Algorithms that are (locally) ISS to disturbances:

[HKC13], [CDS20] Newton-like methods for equation systems

[LMNK20] a class of q-linearly convergent algorithms for optimal control

[Son22] gradient descent with Polyak-Łojasiewicz (PL) condition

[CK23] proximal gradient descent with strong convexity or PL

[dOSS23] Newton's method for gradient systems

[CK24b] Josephy-Newton methods for generalized equations including SQP and augmented Lagrangian methods

Brief Introduction: Variational Analysis

Consider the problem

$$\min_x \varphi(x) \quad \text{subject to } x \in C \tag{P}$$

with $\varphi : X \to \mathbb{R}$ continuously (Fréchet) differentiable
and $C \subset X$ closed and convex.

Let $\bar{x}$ be a *local minimum*

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- that is, for a neighbourhood U of $\bar{x}$,

$$\forall x \in C \cap U, \quad \varphi(x) - \varphi(\bar{x}) \geq 0$$

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$$\forall x \in C \cap U, \quad \varphi(x) - \varphi(\bar{x}) \geq 0$$

- then [Don21]

$$\forall x \in C, \quad \nabla\varphi(\bar{x})(x - \bar{x}) \geq 0 \tag{VI}$$

Brief Introduction: Generalized Equations

The necessary conditions (VI) are equivalent to

$$\nabla\varphi(\bar{x}) + N(\bar{x}, C) \ni 0 \tag{GE}$$

where $N(\cdot, C) : x \mapsto \mathcal{N}_x \subset X^*$ is the normal cone mapping.

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- $\left[\nabla\varphi + N(\cdot, C)\right] : X \rightrightarrows X^*$ is a *set-valued mapping* (SVM)

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- $\left[\nabla\varphi + N(\cdot, C)\right] : X \rightrightarrows X^*$ is a *set-valued mapping* (SVM)
- similarly, Karush–Kuhn–Tucker (KKT) conditions with $y \in Y^*$ can be written as $F : X \times Y^* \rightrightarrows X^* \times Y$
- nonlinear optimization algorithms often solve (GE) in lieu of (P)

Brief Introduction: Newton Methods

1. Primal problem and its necessary conditions

$$\min_x \varphi(x) \quad \text{s.t. } x \in C \qquad\qquad \nabla\varphi(x) + N(x, C) \ni 0$$

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$$\min_x \varphi(x) \quad \text{s.t. } x \in C \qquad\qquad \nabla\varphi(x) + N(x, C) \ni 0$$

2. Approximate at $x_k \in X$

$$\min_x \tfrac{1}{2}\nabla^2\varphi(x_k)(x - x_k, x - x_k) + \nabla\varphi(x_k)(x - x_k) \quad \text{s.t. } x \in C$$

$$\nabla\varphi(x_k) + \nabla^2\varphi(x_k)(x - x_k) + N(x, C) \ni 0$$

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3. Solve for next iterate

$$x_{k+1} \in \left[\nabla^2\varphi(x_k) + N(\cdot, C)\right]^{-1}\left(\nabla^2\varphi(x_k)x_k - \nabla\varphi(x_k)\right)$$

and repeat.

Brief Introduction: Regularity of SVMs

Newton methods for optimization:

- solve $F(x) \ni 0$ through iteration $x_{k+1} \in \Phi(x_k)$

Idea of (strong or metric) regularity: F and Φ behave 'nicely' around $\bar{x}$

Remark

Notions of *regularity* include (imply)

1. surjectivity or openness
2. injectivity
3. nonsingular linear operator

Brief Introduction: Regularity of SVMs

Newton methods for (perturbed) optimization:

- solve $F(x, v) \ni 0$ through iteration $x_{k+1} \in \Phi(x_k, v_k)$

Idea of (strong or metric) regularity: F and Φ behave 'nicely' around $\bar{x}$

Remark

Notions of *regularity* include (imply)

1. surjectivity or openness
2. injectivity
3. nonsingular linear operator

and these notions are stable under perturbation v.

Outline

Optimization algorithms under strong regularity

Strong regularity in nonlinear optimization

Systems-theoretical characterization of strong regularity

Problem Setup

Consider the nonlinear optimization problem

$$\min_x \varphi(x) \quad \text{s.t.} \;\; g(x) = 0 \text{ and } x \in \Omega \tag{1}$$

with $\varphi : X \to \mathbb{R}$, $g : X \to Y$, and $\Omega \subseteq X$

Assume that

1. X is Asplund and $\bar{x} \in X$ is local optimal solution
2. φ and g are continuously Fréchet differentiable
   (a fortiori strictly differentiable) around $\bar{x}$
3. $\Omega \subseteq X$ is nonempty, closed, and convex
4. either Ω or $\{0\}$ is SNC at $\bar{x}$ or 0, respectively

Problem Setup

Consider the nonlinear optimization problem

$$\min_x \varphi(x) \quad \text{s.t.} \;\; g(x) = 0 \text{ and } x \in \Omega \tag{1}$$

with $\varphi : X \to \mathbb{R}$, $g : X \to Y$, and $\Omega \subseteq X$

- The *Karush–Kuhn–Tucker* (KKT) necessary conditions for (1) are

$$F(x, y) = \begin{pmatrix} \nabla\varphi(x) + \nabla g(x)^* y \\ g(x) \end{pmatrix} + \begin{bmatrix} N(x, \Omega) \\ \{0\} \end{bmatrix} \ni 0 \tag{2}$$

  with duals $y \in Y^*$ and normal cone $N(\cdot, \Omega) : X \rightrightarrows X^*$

Example

A *discrete-time optimal control problem* is given as

$$\min_{x = \binom{\xi_1,\ldots,\xi_N}{\upsilon_0,\ldots,\upsilon_{N-1}}} \underbrace{\sum_{k=0}^{N} \ell(\xi_k, \upsilon_k)}_{=\varphi(x)} \quad \text{s.t.} \quad \underbrace{\begin{pmatrix} \xi_1 - \psi(\xi_0, \upsilon_0) \\ \vdots \\ \xi_N - \psi(\xi_{N-1}, \upsilon_{N-1}) \end{pmatrix}}_{=g(x)} = 0 \text{ and } x \in \underbrace{\mathcal{X} \times \mathcal{U}}_{=\Omega}$$

where $\mathcal{X}$ and $\mathcal{U}$ are state and input constraint sets (polygonal or hyperboxes)

This problem is a *nonlinear program* (NLP):

- X and Y are finite-dimensional
- φ and g are (usually) twice differentiable
- Ω is given by linear constraints

Generalized Newton Methods

Solve the KKT generalized equation

$$\underbrace{\begin{pmatrix} \nabla\varphi(x) + \nabla g(x)^* y \\ g(x) \end{pmatrix}}_{= f(z)} + \underbrace{\begin{bmatrix} N(x,\Omega) \\ \{0\} \end{bmatrix}}_{= N(z)} \ni 0$$

via the iteration

$$f(z_k) + H(z_k)(z_{k+1} - z_k) + N(z_{k+1}) \ni 0 \tag{3}$$

for a suitable operator $H(\cdot)$ and $z = (x, y)$

Generalized Newton Methods

Solve the KKT generalized equation

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Remark (Sequential quadratic programming)

$$H(z) = \begin{pmatrix} \nabla^2(\varphi(x) + \langle g(x), y\rangle) & \nabla g(x)^* \\ \nabla g(x) & 0 \end{pmatrix}$$

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Solve the KKT generalized equation

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Remark (Sequential linear programming)

$$H(z) = \begin{pmatrix} 0 & \nabla g(x)^* \\ \nabla g(x) & 0 \end{pmatrix}$$

Generalized Newton Methods

Solve the KKT generalized equation

$$\underbrace{\begin{pmatrix} \nabla\varphi(x) + \nabla g(x)^* y \\ g(x) \end{pmatrix}}_{= f(z)} + \underbrace{\begin{bmatrix} N(x, \Omega) \\ \{0\} \end{bmatrix}}_{= N(z)} \ni 0$$

via the iteration

$$f(z_k) + H(z_k)(z_{k+1} - z_k) + N(z_{k+1}) \ni 0 \tag{3}$$

Remark (Projected gradient)

$$H(z) = \begin{pmatrix} \alpha^{-1}\mathbb{I} & 0 \\ 0 & \alpha^{-1}\mathbb{I} \end{pmatrix}$$

Perturbed Newton Methods

Solve the *perturbed* generalized equation

$$f(z, v) + N(z) \ni 0$$

via the perturbed generalized Newton iteration

$$z_{k+1} \in \Phi_H(z_k, v_k) \iff_{\text{def}} f(z_k, v_k) + H(z, v_k)(z_{k+1} - z_k) + N(z_{k+1}) \ni 0$$

for a suitable operator $H(\cdot)$

Strong Regularity

Definition

F is *strongly regular* at $\bar{z}$ for $\bar{v} \in F(\bar{z})$ iff, with neighbourhoods U of $\bar{z}$ and V of $\bar{v}$,

$$\forall v \in V, \quad F^{-1}(v) \cap U = \{s(v)\}$$

and $s(\cdot)$ is Lipschitz continuous around $\bar{v}$.

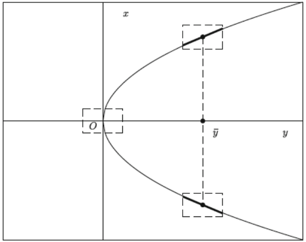

Figure: The inverse of $z \mapsto z^2$.

Equivalent: [Don21]

1. F is strongly regular at $\bar{z}$ for $\bar{v}$,
2. F^{-1} has Lipschitz continuous, single-valued localization at $\bar{v}$ for $\bar{z}$
3. F is linearly open (a fortiori surjective) and locally injective at $\bar{z}$ for $\bar{v}$

ISS of Newton Methods

Let $f_H(z', z, v) = f(z, v) + H(z, v)(z' - z)$;

Theorem ([CK24b])

Suppose that

1. $\bar{z}$ *is a solution of* $f(\cdot, 0) + N \ni 0$
2. f_H *is uniformly Lipschitz continuous (constants* γ_z *and* γ_v*) at* $(\bar{z}, \bar{z}, 0)$
3. $f_H(\cdot, \bar{z}, 0) + N$ *is strongly regular (constant* κ*) at* $\bar{z}$ *for* 0

and $\kappa\gamma_z < 1$*; then the iteration* $z_{k+1} \in \Phi(z_k, v)$
is locally unique and locally input-to-state stable.

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and $\kappa\gamma_z < 1$*; then the iteration* $z_{k+1} \in \Phi(z_k, v)$
is locally unique and locally input-to-state stable.

Proof sketch: The update has a locally unique solution $s(\cdot)$ with

$$\|z_{k+1} - \bar{z}\| = \|s(z_k, v_k) - s(\bar{z}, 0)\| \leq \kappa\gamma_z \|z_k - \bar{z}\| + \kappa\gamma_v \|v_k\|$$

Generalized Implicit Function Theorem

Let $f_H(z', p) = f(p) + H(p)(z' - p_1)$ with $p = (z, v)$;

Proposition ([Don21])

Suppose that

1. $\bar{z}$ *is a solution of* $f_H(\cdot, \bar{p}) + N \ni 0$
2. f_H *is uniformly Lipschitz continuous (constant* γ_p*) at* $(\bar{z}, \bar{p})$
3. $f_H(\cdot, \bar{p}) + N$ *is strongly regular (constant* κ*) at* $\bar{z}$ *for* 0

then

$$S : p \mapsto \{z \in X \times Y^* \mid f_H(z, p) + N(z) \ni 0\}$$

has a Lipschitz continuous (constant $\kappa\gamma_p$*) and single-valued localization* $s(\cdot)$ *at* $\bar{p}$ *for* $\bar{z}$*.*

Strong Regularity in Nonlinear Optimization

The mapping

$$f_H(\cdot, \bar{p}) + N : (x, y) \mapsto f(\bar{p}) + \begin{bmatrix} H_{xx} & H_{yx}^* \\ H_{yx} & 0 \end{bmatrix} \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix} + N(x, \Omega)$$

with $H_{xx} \succeq 0$ is strongly regular at $\bar{z}$ for 0 *if and only if*

$$\min_x H_{xx}(x - \bar{x}, x - \bar{x}) + [\nabla\varphi(\bar{x}) - \delta_x](x - \bar{x})$$
$$\text{s.t. } g(\bar{x}) + H_{yx}(x - \bar{x}) = \delta_y \text{ and } x \in \Omega$$

has a unique primal-dual solution (x_δ, y_δ) for $\delta = (\delta_x, \delta_y)$ close to 0 with

$$\|(x_{\delta 1}, y_{\delta 1}) - (x_{\delta 2}, y_{\delta 2})\| \leq \kappa \|\delta_1 - \delta_2\|$$

Nonlinear Programs revisited

Consider the nonlinear program

$$\min_x \varphi(x) \quad \text{s.t.} \;\; g(x) = 0 \text{ and } x \in \Omega = \mathbb{R}^n_{\geq 0} \tag{4}$$

with $\varphi : X \to \mathbb{R}$, $g : X \to Y$, and $\Omega \subseteq X$

Assume that

1. X and Y are finite-dimensional; and $\bar{x} \in X$ is local optimal solution
2. φ and g are *twice* continuously Fréchet differentiable
3. $\Omega = \mathbb{R}^n_{\geq 0}$ is the nonnegative orthant

Note: $N(x, \mathbb{R}^n_{\geq 0}) \subseteq \mathbb{R}^n_{\leq 0}$

Constraint Qualifications

MFCQ

If the constraint qualification

$$\{\nabla g(\bar{x})^* y \,|\, y \in Y^*\} \cap \big[-N(\bar{x}, \mathbb{R}^n_{\geq 0})\big] = \{0\}$$

holds and $\nabla g(\bar{x})$ is surjective, then there exists $\bar{y}$ with $F(\bar{x}, \bar{y}) = 0$.

LICQ

If the active constraints

$$g_0 : x \mapsto (x_{i \in I_0}, g(x)), \quad \text{where } x_i = 0 \Leftrightarrow i \in I_0,$$

have a surjective $\nabla g_0(\bar{x})$, then there exists a unique $\bar{y}$ with $F(\bar{x}, \bar{y}) = 0$.

Strong Stability in Nonlinear Programs

$\bar{x} \in X$ is a *stationary solution* if and only if $F(\bar{x}, \bar{y}) = 0$ for some $\bar{y} \in Y^*$.

Definition ([Koj80])

A stationary solution $\bar{x}$ is *strongly stable* if and only if there exists a neighbourhood U of $\bar{x}$ and $d > 0$ such that

$$\min_{x} \varphi(x) + \langle \Delta x + \delta_\varphi, x \rangle \quad \text{s.t. } g(x) = \delta_y \text{ and } x + \delta_x \in \mathbb{R}^n_{\geq 0}$$

has a unique stationary solution $s(\cdot) \in U$ for $\|(\Delta, \delta_\varphi, \delta_y, \delta_x)\| \leq d$ which is continuous at 0.

Strong Stability & Strong Regularity

If F is strongly regular at $(\bar{x}, \bar{y})$ for 0, then

- $\bar{x}$ is a strongly stable stationary solution
- MFCQ holds at $\bar{x}$ with unique dual $\bar{y}$

Strong Stability & Strong Regularity

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An *optimal* solution $\bar{x}$ is a strongly stable stationary solution if and only if

- MFCQ holds
- the strong second-order sufficient condition is satisfied
  (a fortiori, $\bar{x}$ is a strict local minimum)

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An *optimal* solution $\bar{x}$ is a strongly stable stationary solution if and only if

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If $\bar{x}$ is a strongly stable stationary solution and LICQ holds, then

- F is strongly regular at $(\bar{x}, \bar{y})$ for 0

Strong Regularity beyond Nonlinear Programs

Consider again

$$\min_x \varphi(x) \quad \text{s.t.} \;\; g(x) = 0 \text{ and } x \in \Omega \tag{1}$$

with $\varphi : X \to \mathbb{R}$, $g : X \to Y$ (continuously differentiable, nonsmooth), and $\Omega \subseteq X$ (nonempty closed convex)

Conjecture (for suitable X and $\|\cdot\|$)

The KKT system of (1) is strongly regular at $(\bar{x}, \bar{y})$ for 0 if and only if

1. *MFCQ holds with unique duals under some perturbations,*
2. *the nonsmooth strong second-order sufficient condition*

$$\forall x \in \Omega \cap U, \quad \varphi(x) + \langle \bar{y}, g(x) \rangle + \frac{\varrho}{2}\|g(x)\|^2 \geq \varphi(\bar{x}) + \frac{\mu}{2}\|x - \bar{x}\|^2$$

is satisfied for some $\varrho, \mu > 0$ and neighbourhood $U \subset X$.

Towards a Systems-theoretical Characterization

Consider the *canonically perturbed* optimization problem

$$\min_x \varphi(x) - \langle v_x, x \rangle \quad \text{s.t. } g(x) = v_y \text{ and } x \in \Omega \tag{5}$$

with $\varphi : X \to \mathbb{R}$, $g : X \to Y$, and $\Omega \subseteq X$ for $(v_x, v_y) \in X^* \times Y$

- The KKT conditions for (5) become

$$F(x,y) = \underbrace{\begin{pmatrix} \nabla\varphi(x) + \nabla g(x)^* y \\ g(x) \end{pmatrix}}_{= f(z)} + \underbrace{\begin{bmatrix} N(x,\Omega) \\ \{0\} \end{bmatrix}}_{= N(z)} \ni \underbrace{\begin{pmatrix} v_x \\ v_y \end{pmatrix}}_{= v} \tag{6}$$

with duals $y \in y$ and normal cone $N(\cdot, \Omega) : X \rightrightarrows X^*$

Necessary Conditions for ISS

Suppose that the generalized Newton's iteration

$$z_{k+1} \in \Phi_H(z_k, v_k) \iff f(z_k) + H(z_k)(z_{k+1} - z_k) - v_k + N(z_k) \ni 0$$

is locally input-to-state stable around $\bar{z} \in F^{-1}(0)$, that is,

$$\|z_N - \bar{z}\| \leq \alpha^N \|z_0 - \bar{z}\| + \gamma \sup_{k \geq 0} \|v_k\|$$

for all $z_0 \in U$, $z_{k+1} \in \Phi_H(z_k, v_k) \cap U$, $v_k \in V$, and $N \geq k \geq 0$,
where $\alpha \in (0, 1)$ and $\gamma \geq 0$

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$$\|z_N - \bar{z}\| \leq \alpha^N \|z_0 - \bar{z}\| + \gamma \sup_{k \geq 0} \|v_k\|$$

- Any fixpoint $z_v \in \Phi_H(z_v, v) \cap U$ for $v \in V$ satisfies

$$\|z_v - \bar{z}\| \leq \gamma \|v\|$$

Necessary Conditions for ISS

Suppose that the generalized Newton's iteration

$$z_{k+1} \in \Phi_H(z_k, v_k) \iff f(z_k) + H(z_k)(z_{k+1} - z_k) - v_k + N(z_k) \ni 0$$

is locally input-to-state stable around $\bar{z} \in F^{-1}(0)$, that is,

$$\|z_N - \bar{z}\| \leq \alpha^N \|z_0 - \bar{z}\| + \gamma \sup_{k \geq 0} \|v_k\|$$

- Any solution $z_v \in (f + N)^{-1}(v) \cap U$ for $v \in V$ satisfies

$$\|z_v - \bar{z}\| \leq \gamma \|v\|$$

Sufficient Conditions for Strong Subregularity

Suppose that the generalized Newton's iteration

$$z_{k+1} \in \Phi_H(z_k, v_k) \iff f(z_k) + H(z_k)(z_{k+1} - z_k) - v_k + N(z_k) \ni 0$$

is locally input-to-state stable around $\bar{z} \in F^{-1}(0)$, that is,

$$\|z_N - \bar{z}\| \leq \alpha^N \|z_0 - \bar{z}\| + \gamma \sup_{k \geq 0} \|v_k\|$$

- Any solution $z_v \in (f + N)^{-1}(v) \cap U$ for $v \in V$ satisfies

$$\|z_v - \bar{z}\| \leq \gamma \|v\|$$

- Hence, $f + N$ is *strongly subregular* at $\bar{z}$ for 0

Sufficient Conditions for Strong Regularity

Theorem (work in progress)

If the generalized Newton's iteration

$$z_{k+1} \in \Phi_H(z_k, v_k) \iff f(z_k) + H(z_k)(z_{k+1} - z_k) - v_k + N(z_k) \ni 0$$

1. *has a fix point* $z_v \in \Phi_H(z_v, v) \cap U$ *for all* $v \in V$ *and*
2. *is* *locally* incrementally *ISS* *around* $\bar{z} \in F^{-1}(0)$, *that is,*

$$\|z'_N - z_N\| \leq \alpha^N \|z'_0 - z_0\| + \gamma \sup_{k \geq 0} \|v'_k - v_k\|$$

for all $z_0^{(\prime)} \in U$, $z_{k+1}^{(\prime)} \in \Phi_H(z_k^{(\prime)}, v_k^{(\prime)}) \cap U$, $v_k^{(\prime)} \in V$, $N \geq k \geq 0$, *where* $\alpha \in (0,1)$ *and* $\gamma \geq 0$,

Then $f + N$ *is strongly regular at* $\bar{z}$ *for 0.*

Concluding Remarks

Regularity of the KKT conditions

- impacts sensitivity and stability in nonlinear optimization
- relates to stable stationary solutions and second-order sufficiency conditions in nonlinear programs (NLP) *and beyond*
- applies to nonlinear optimization problems other than NLPs e.g., nonconvex semidefinite or sum-of-squares programs

Beyond Strong Regularity

- Strong regularity implies local incremental ISS and uniqueness under perturbations

Alternatives:

1. Strong subregularity (implies ISS)
2. Metric regularity (implies existence)

Remark

Strong regularity and metric regularity are equivalent for NLPs.

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Thank you!

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Nonlinear Sum-of-squares Optimization

A nonlinear polynomial program is

$$\min_{\xi} \varphi(\xi) \quad \text{s.t.} \;\; g(\xi) = 0 \text{ and } \xi \in \Sigma[x] \tag{7}$$

with $\varphi : \mathbb{R}[x] \to \mathbb{R}$, $g : \mathbb{R}[x] \to \mathbb{R}[x]$, and sum-of-squares cone $\Sigma[x] \subset \mathbb{R}[x]$

- these problems arise in analysis and control synthesis of nonlinear dynamic systems, e.g.,

$$\begin{aligned} &\min \int_{\mathcal{R}} [V(x) - h(x)]^2 \\ &\text{s.t.} \;\; s(x)\,[V(x) - 1] - \nabla V(x)\psi(x) - \varepsilon\|x\|^2 \in \Sigma[x] \\ &\text{and} \;\; V(x) - \varepsilon\|x\|^2 \in \Sigma[x] \text{ and } s(x) \in \Sigma[x] \end{aligned}$$

Nonlinear Sum-of-squares Optimization

A nonlinear polynomial program is

$$\min_{\xi} \varphi(\xi) \quad \text{s.t.} \;\; g(\xi) = 0 \text{ and } \xi \in \Sigma[x] \tag{7}$$

with $\varphi : \mathbb{R}[x] \to \mathbb{R}$, $g : \mathbb{R}[x] \to \mathbb{R}[x]$, and sum-of-squares cone $\Sigma[x] \subset \mathbb{R}[x]$

- generalized Newton's iteration takes the form of a *convex* sum-of-squares problem
- this iteration is asymptotically convergent under strong regularity

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