Nonlinear Sum-of-squares Optimization
Theory, Methods, Applications

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Evolution of Flight Control Algorithms

1903

human control

1988

fly-by-wire[a]

[a] Laurent ERRERA, CC BY-SA 2.0

2009

MPC[a]

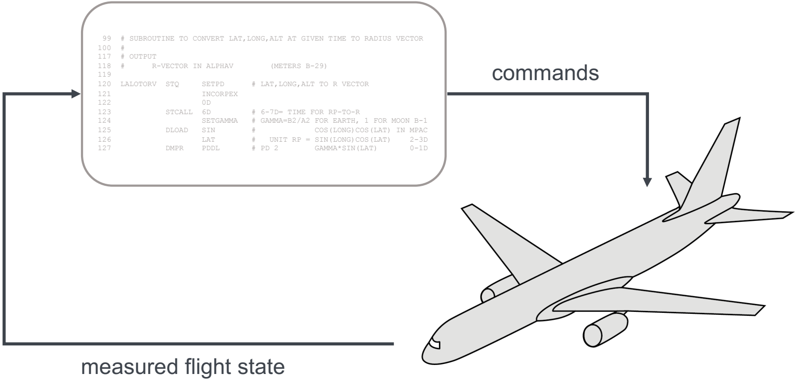
[a] 2009 de Almeida & Leißling

2024

AI pilot[a]

[a] Damian Dovarganes/AP

Algorithms as Feedback



Global Stability and the Region of Attraction

Global Stability

A dynamic system $\dot{x} = f(x)$ is *globally asymptotically stable* if and only if

$$\lim_{t\to\infty} \|x(t)\| = 0$$

for all $x(0) = x_0 \in \mathbb{R}^n$

Region of Attraction

A dynamic system $\dot{x} = f(x)$ is *asymptotically stable* on $\mathcal{R} \subset \mathbb{R}^n$ if and only if

$$x_0 \in \mathcal{R} \Rightarrow \lim_{t\to\infty} \|x(t)\| = 0$$

and $x(\cdot) \subset \mathcal{R}$ for all $x(0) = x_0 \in \mathbb{R}^n$

Zubov's Equation

Proposition ([Zub64], cited after [GTV85])

Let $V : \mathbb{R}^n \to R$ be a Lyapunov candidate function; if

$$\langle \nabla V(x), f(x) \rangle = \phi(x) \left[V(x) - 1 \right] \tag{1}$$

for all $x \in \mathbb{R}^n$, where $\phi : \mathbb{R}^n \to \mathbb{R}$ is positive definite, then

$$\mathcal{R} = \{ x \in \mathbb{R}^n \mid V(x) < 1 \}$$

is the region of attraction.

Relaxing Zubov's Equation

Let $V : \mathbb{R}^n \to R$ be a Lyapunov candidate function; if

$$\langle \nabla V(x), f(x) \rangle \leq \phi(x) \, [V(x) - 1] \tag{2}$$

for all $x \in \mathbb{R}^n$, where $\phi : \mathbb{R}^n \to \mathbb{R}$ is positive definite, then

$$\hat{\mathcal{R}} = \{x \in \mathbb{R}^n \mid V(x) < 1\} \subseteq \mathcal{R}$$

is an *invariant subset* of the region of attraction.

Sum-of-squares Polynomials

Definition

A polynomial $f \in \mathbb{R}_{2d}[x]$ is *sum-of-squares* ($f \in \Sigma[x]$) if and only if

$$f = \sum_{i=1}^{n_d} (f_i)^2$$

with $f_1, \ldots, f_{n_d} \in \mathbb{R}_d[x]$.

Note: $f \in \Sigma[x] \Rightarrow \forall x \in \mathbb{R}^n,\ f(x) \geq 0$

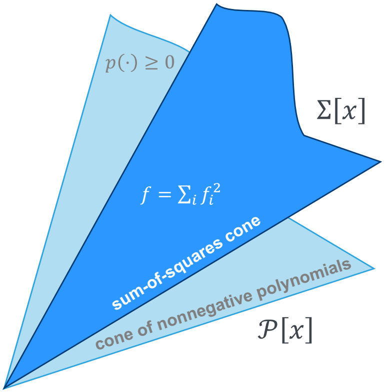


Figure: Cones $\Sigma[x]$ and $\mathcal{P}[x]$ illustrated.

Sum-of-squares Programming

Zubov's equation can further be relaxed to

$$\phi(x)\,[V(x) - 1] - \langle \nabla V(x), f(x) \rangle \in \Sigma[x]$$

which is a polynomial expression if $V, \phi, f \in \mathbb{R}[x]$

Applications

- region of attraction analysis [CSB11]
- dissipation-based analysis and design [EA06]
- synthesis of control and observer [JWFT$^+$03, Tan06]
- control Lyapunov / barrier functions [PJ04, TP04, ACE$^+$19]

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most of which include nonlinear constraints

Nonlinear Sum-of-squares Optimization

A nonlinear polynomial optimization problem is

$$\min_{\xi} \varphi(\xi) \quad \text{s.t.} \;\; g(\xi) \in \Sigma[x] \text{ and } \xi \in \Sigma[x] \tag{3}$$

with $\varphi : \mathbb{R}[x] \to \mathbb{R}$, $g : \mathbb{R}[x] \to \mathbb{R}[x]$, and sum-of-squares cone $\Sigma[x] \subset \mathbb{R}[x]$

- these problems arise in analysis and control of nonlinear dynamic systems, e.g.,

$$\begin{aligned} &\min \int_{\mathcal{R}} [V(x) - h(x)]^2 \\ &\text{s.t.} \;\; s(x)\,[V(x) - 1] - \langle \nabla V(x), f(x) \rangle - \varepsilon \|x\|^2 \in \Sigma[x] \\ &\text{and} \;\; V(x) - \varepsilon \|x\|^2 \in \Sigma[x] \text{ and } s(x) \in \Sigma[x] \end{aligned}$$

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- generalized Newton's method takes the form of *convex* sum-of-squares problems
- this iteration is asymptotically convergent under strong regularity

Sequential Sum-of-squares Programming

Solve the (primal) nonlinear problem

$$\min_{\xi} \varphi(\xi) \quad \text{s.t.} \;\; g(\xi) \in \Sigma[x] \text{ and } \xi \in \Sigma[x] \tag{3}$$

via an iteration of convex approximations

$$\min_{\nu} \langle H(\xi_k)\nu, \nu\rangle + \nabla\varphi(\xi_k)\nu \\ \text{s.t.} \;\; g(\xi_k) + \nabla g(\xi_k)\nu \in \Sigma[x] \text{ and } \xi_k + \nu \in \Sigma[x] \tag{4}$$

for some suitable operator $H(\cdot) : \mathbb{R}[x] \to \mathbb{R}[x]^*$

Necessary Conditions for Sum-of-squares

The nonlinear *Karush-Kuhn-Tucker* (KKT) conditions are

$$\underbrace{\begin{pmatrix} \nabla\varphi(\xi) + \nabla g(\xi)^* \eta \\ g(\xi) \end{pmatrix}}_{= f(z)} + \underbrace{\begin{bmatrix} N(\xi, \Sigma[x]) \\ N(\eta, \Sigma[x]^*) \end{bmatrix}}_{= F(z)} \ni 0 \tag{5}$$

and the convex KKT conditions are

$$\underbrace{\begin{pmatrix} \nabla\varphi(\xi_k) \\ g(\xi_k) \end{pmatrix}}_{= f(z_k)} + \underbrace{\begin{bmatrix} H(\xi_k) & \nabla g(\xi_k)^* \\ \nabla g(\xi_k) & 0 \end{bmatrix}}_{= \mathcal{H}(z_k) \approx \nabla f(z_k)} \begin{pmatrix} \xi - \xi_k \\ \eta - \eta_k \end{pmatrix} + \underbrace{\begin{bmatrix} N(x, \Omega) \\ \{0\} \end{bmatrix}}_{= F(z)} \ni 0 \tag{6}$$

Brief Introduction: Newton Methods

1. Primal problem and its necessary conditions

$$\min_x \varphi(x) \quad \text{s.t. } x \in C \qquad\qquad \nabla\varphi(x) + N(x, C) \ni 0$$

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2. Approximate at x_k

$$\min_x \tfrac{1}{2}\nabla^2\varphi(x_k)(x - x_k, x - x_k) + \nabla\varphi(x_k)(x - x_k) \quad \text{s.t. } x \in C$$

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$$\nabla\varphi(x_k) + \nabla^2\varphi(x_k)(x - x_k) + N(x, C) \ni 0$$

3. Solve for next iterate

$$x_{k+1} \in \left[\nabla^2\varphi(x_k) + N(\cdot, C)\right]^{-1}\left(\nabla^2\varphi(x_k)x_k - \nabla\varphi(x_k)\right)$$

and repeat.

Newton Methods for Nonlinear Optimization

1. Necessary conditions

$$f(z) + F(z) \ni 0$$

2. Approximate at z_k

$$f(z_k) + \mathcal{H}(z_k)(z - z_k) + F(z) \ni 0$$

3. Solve for next iterate

$$z_{k+1} \in \Phi(z_k) := \left[\mathcal{H}(z_k) + F\right]^{-1}\left(\mathcal{H}(z_k) z_k - f(z_k)\right)$$

Strong Regularity

Definition

$f + F$ is *strongly regular* at $\bar{z}$ for 0 iff, with neighbourhoods U of $\bar{z}$ and V of 0,

$$\forall v \in V, \quad F^{-1}(v) \cap U = \{s(v)\}$$

and $s(\cdot)$ is Lipschitz continuous around 0.

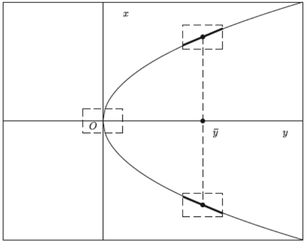

Figure: The inverse of $z \mapsto z^2$.

Equivalent: [Don21]

1. F is strongly regular at $\bar{z}$ for $\bar{v}$,
2. F^{-1} has Lipschitz continuous, single-valued localization at $\bar{v}$ for $\bar{z}$
3. F is linearly open (a fortiori surjective) and locally injective at $\bar{z}$ for $\bar{v}$

Asymptotic Convergence of Newton Methods

Let $f_H(z', z) = f(z) + \mathcal{H}(z)(z' - z)$;

Theorem ([CL23, CK24])

Suppose that

1. $\bar{z}$ *is a solution of* $f + F \ni 0$
2. $f_H(z', \cdot)$ *is uniformly Lipschitz continuous (constants* γ*) at* $(\bar{z}, \bar{z})$
3. $f_H(\cdot, \bar{z}) + F$ *is strongly regular (constant* κ*) at* $\bar{z}$ *for* 0

and $\kappa\gamma < 1$*; then the iteration* $z_{k+1} \in \Phi(z_k)$
is locally unique and locally asymptotically stable.

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is locally unique and locally asymptotically stable.

Proof sketch: The update has a locally unique solution $s(\cdot)$ with

$$\|z_{k+1} - \bar{z}\| = \|s(z_k) - s(\bar{z})\| \leq \kappa\gamma_z \|z_k - \bar{z}\|$$

Strong Regularity in Nonlinear Optimization

The mapping

$$f_H(\cdot, \bar{z}) + F : (\xi, \eta) \mapsto f(\bar{z}) + \begin{bmatrix} H & \nabla g^*(\bar{\xi}) \\ \nabla g(\bar{\xi}) & 0 \end{bmatrix} \begin{pmatrix} \xi - \bar{\xi} \\ \eta - \bar{\eta} \end{pmatrix} + F(z)$$

with $H \succeq 0$ is strongly regular at $\bar{z}$ for 0 *if and only if*

$$\min_x H(\xi - \bar{\xi}, \xi - \bar{\xi}) + [\nabla \varphi(\bar{\xi}) - d_\xi](\xi - \bar{\xi})$$
$$\text{s.t.} \;\; g(\bar{\xi}) + \nabla g(\bar{\xi})(\xi - \bar{\xi}) \in d_\eta + \Sigma[x] \text{ and } \xi \in \Sigma[x]$$

has a unique primal-dual solution (ξ_d, η_d) for $d = (d_\xi, d_\eta)$ close to 0 with

$$\|(\xi_{d1}, \eta_{d1}) - (\xi_{d2}, \eta_{d2})\| \leq \kappa \|d_1 - d_2\|$$

Globalization Techniques

- Trust-region techniques: to ensure that $\xi_{k+1} \in \xi_k + U$,
  1. add constraint $\|\xi - \xi_k\| \leq \varrho$
  2. add regularization $\frac{\varrho}{2}\|\xi - \xi_k\|^2$

- Line-search methods: find new solution $\xi_{k+1} = (1-\alpha)\xi_k + \alpha\xi^*$ subject to $\alpha \in (0, 1]$ and
  1. minimizes merit function

$$\phi(\xi_\alpha) := \varphi(\xi_\alpha) + \frac{\varrho}{2}\mathrm{viol}(\xi_\alpha)^2$$

  2. accepted by filter if

$$\varphi(\xi_\alpha) < \varphi(\xi_k) \wedge \mathrm{viol}(\xi_\alpha) < \mathrm{viol}(\xi_k)$$

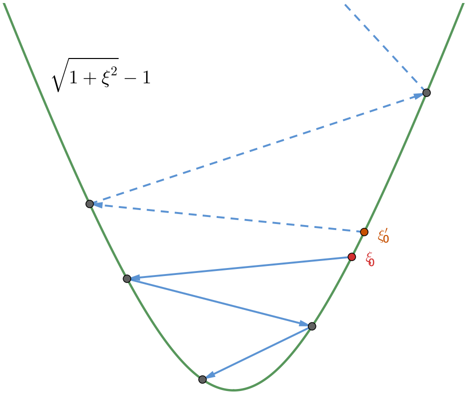


Figure: Newton iteration.

Constraint Violation

Goal of $\mathrm{viol}(\xi_0)$: measure distance of $g(\xi_0)$ to sum-of-squares cone

1. distance function

$$\min_{\gamma} \frac{1}{2}\|\gamma - g(\xi_0)\|^2 \quad \text{s.t. } \gamma \in \Sigma[x]$$

2. signed distance

$$\min\{r \in \mathbb{R} \quad \text{s.t. } g(\xi_0) + r\,\vec{\gamma} \in \Sigma[x]\}$$

where $\vec{\gamma} \in \operatorname{int}\Sigma[x]$

Example: Region of Attraction Estimation

- find Lyapunov candidate $V(\cdot)$ with

$$V(x) \leq 1 \Rightarrow \langle \nabla V(x), f(x) \rangle < 0$$

for all $x \in \mathbb{R}^n$ (here $n = 4$)

- Optimal solution found after 6 iterations

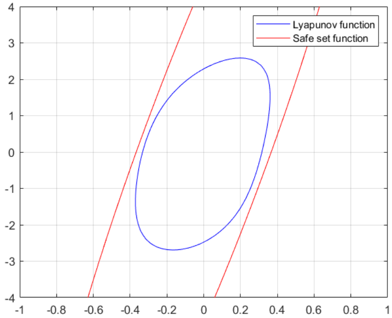


Figure: Region-of-attraction estimate for GTM longitudinal motion.

Example: Nonlinear Control Synthesis

- find Lyapunov candidate $V(\cdot)$ and feedback $k(\cdot)$ with

$$V(x) \leq 1 \Rightarrow \langle \nabla V(x), f(x, k(x)) \rangle < 0$$

for all $x \in \mathbb{R}^n$ (here $n = 4$)

- Optimal solution found after 9 iterations

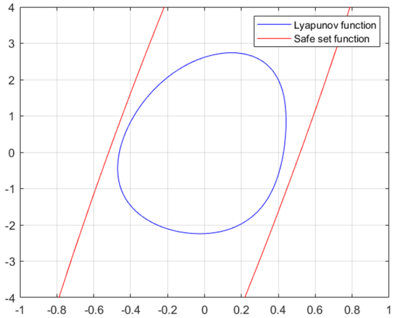


Figure: Nonlinear control synthesis for GTM longitudinal motion.

Remark on Quadratic Cost Functions

- we often want to 'maximize' a semialgebraic set $\{x \in \mathbb{R}^n \mid V(x) \leq 1\}$ subject to bilinear constraints

- the quadratic cost of the convex approximation then becomes

$$\nabla_V^2 \left[\varphi(V_k) + \langle \eta_k, g(V_k, s_k) \rangle\right] = \nabla_V^2 \varphi(V_k)$$

- Suggestion: use quadratic distance to reference $h \in \mathbb{R}[x]$; viz.

$$\varphi(V) = \int_\Omega (V - h)^2(x)\mathrm{d}x$$

where $\Omega \supseteq \{x \in \mathbb{R}^n \mid h(x) \leq 1\}$

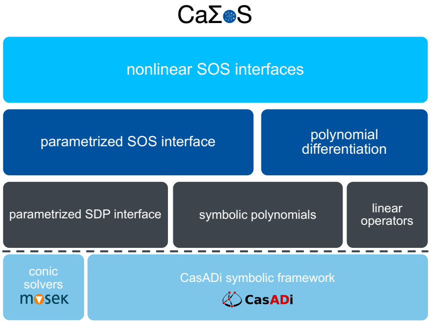
Nonlinear Sum-of-squares Optimization Suite

CaΣoS [CO25]

versatile, optimization-oriented software for convex, quasiconvex, and nonconvex (nonlinear) sum-of-squares problems

$$\min_{\xi} \varphi(\xi) \quad \text{s.t. } \xi \in \mathcal{K}_1,\ g(\xi) \in \mathcal{K}_2$$

- supported cones ($\mathcal{K}_1$, $\mathcal{K}_2$):
  1. sum-of-squares, (S)DSOS
  2. PSD, (S)DD matrices
  3. (rotated) Lorentz, power, exp



https://github.com/ifr-acso/casos

Nonlinear Sum-of-squares Optimization Suite

- CaΣoS significantly reduces the parsing time in repeatedly solved convex sum-of-squares problems

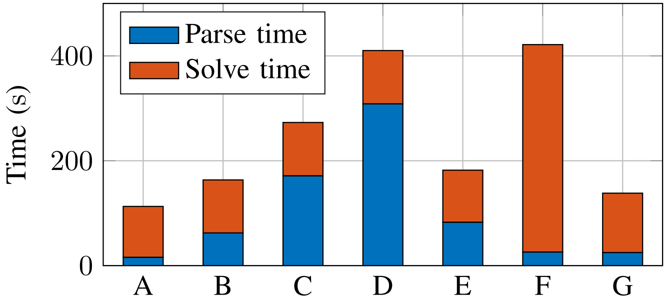


Figure: Comparison of sum-of-squares toolboxes. (A: CaΣoS, B: SOSTOOLS (dpvar), C: SOSTOOLS (pvar), D: SOSOPT, E: SPOTLESS, F: YALMIP, G: SUMOFSQUARES.JL)

Case Study: Model Predictive Control

- *Model Predictive Control* (MPC) solves the control problem

$$
\begin{aligned}
\min_{\mathbf{u},\mathbf{x}} \; & P(x_T) + \sum_{t=0}^{T-1} L(x_t, u_t), \\
\text{s.t.} \; & \left.\begin{aligned} x_{t+1} &= f(x_t, u_t) \\ x_t &\in \mathcal{X} \\ u_t &\in \mathcal{U} \end{aligned}\right\} \; t \in [0, T), \\
& x_T \in \mathcal{X}_T
\end{aligned}
$$

- MPC feedback stabilizes f on the reach-avoid set $\mathcal{R}_{[0,T]}$ [CK21]

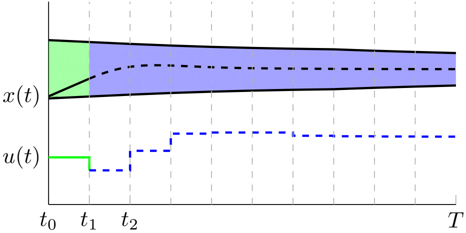


Figure: $\mathcal{R}_{[0,T]}$ is invariant under MPC feedback (illustration from [OFC24]).

Case Study: Horizon-one MPC

Approach [OFC24]

Solve for

1. inner estimate of reach-avoid set

$$\hat{\mathcal{R}} = \{x \in \mathbb{R}^n \mid V(0, x) \leq 0\} \subset \mathcal{R}_{[0,T]}$$

via a dissipation inequality

2. horizon-one MPC feedback

$$\min_{u \in \mathcal{U}} \; \alpha V(1, f(x_0, u)) + L(x_0, u)$$
$$\text{s.t.} \;\; V(1, f(x_0, u)) \leq 0$$

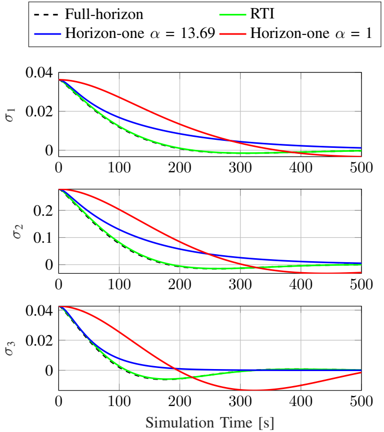


Figure: Comparison of closed-loop responses under MPC approaches.

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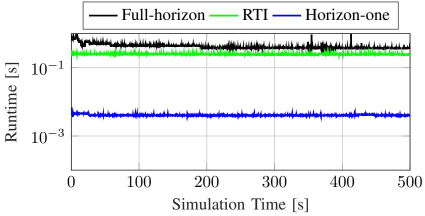

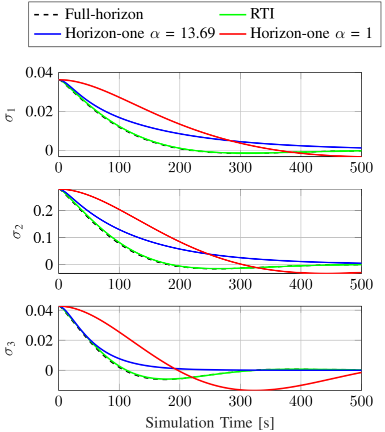

Figure: Comparison of closed-loop responses under MPC approaches.

Concluding Remarks

- sum-of-squares optimization is a powerful tool for nonlinear system analysis and control synthesis
- real-world problems are nonconvex and often large-scale
- nonlinear optimization theory supports sequential methods, decomposition techniques, and inexact optimization

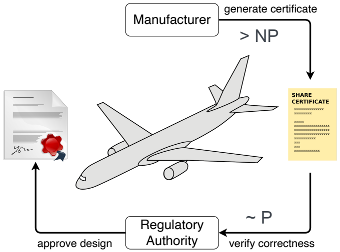

Figure: Computation of SOS certificates is hard, validation is tractable.

Acknowledgments

    

Thank you!

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CaΣoS

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